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On some new modular relations for a remarkable product of theta–functions

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Abstract

In this paper, we establish some new modular equations of degree 9. We also establish several new P–Q mixed modular equations involving theta–functions which are similar to those recorded by Ramanujan in his notebooks. As an application, we establish some new general formulas for explicit evaluations of a Remarkable product of theta–functions.

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1 Introduction

For $|q| < 1$,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty, \tag{1.1}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.2}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{1.3}$$

are special cases of Ramanujan's general theta function [4]

$$\begin{aligned} f(a,b) &:= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ \ |ab| < 1, \\ &= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \end{aligned}$$

where $(a;q)_\infty := \prod_{n=0}^{\infty}(1 - aq^n)$.
At scattered places of his second notebook [18], Ramanujan records a total of nine P–Q mixed modular equations of degrees 1, 3, 5 and 15. These equations were proved by B. C. Berndt and L. -C. Zhang [6] and [7]. S. Bhargava, C. Adiga and M. S. Mahadeva Naika [9] and [10], have established several new P–Q modular equations involving four moduli. For more details on P–Q eta-function identities one can refer [1], [3], [12], [16] and [14].

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In §2, we collect some identities that are needed to prove our main results. In §3, we establish some new modular equations of degree 9. In §4, we establish several new P–Q mixed modular equations akin to those recorded by Ramanujan in his notebooks.

Mahadeva Naika, M. C. Maheshkumar and K. Sushan Bairy [17], have introduced a new remarkable product of theta-functions $b_{s,t}$:

$$b_{s,\,t} = \frac{te^{\frac{-(t-1)\pi}{4}\sqrt{\frac{s}{t}}}\psi^2\left(-e^{-\pi\sqrt{st}}\right)\varphi^2\left(-e^{-2\pi\sqrt{st}}\right)}{\psi^2\left(-e^{-\pi\sqrt{\frac{s}{t}}}\right)\varphi^2\left(-e^{-2\pi\sqrt{\frac{s}{t}}}\right)}, \tag{1.4}$$

where s, t are real numbers such that $s > 0$ and $t \geq 1$. They have established some new general formulas for the explicit evaluations of $b_{s,t}$ and computed some particular values of $b_{s,t}$. Finally in §5, we establish some new modular relations connecting a remarkable product of theta-functions $b_{s,9}$ with $b_{r^2s,9}$ for $r = 2$, 4 and 6 and explicit values of $b_{s,9}$ are deduced.

We end this section by defining a modular equation in brief.
The ordinary or Gaussian hypergeometric function is defined as

$$_2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\,n!} z^n, \quad 0 \leq |z| < 1,$$

and a, b ,c are complex numbers such that $c \neq 0, -1, -2, \ldots$,
where

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1) \ \text{ for } \ n \text{ a positive integer.}$$

Let

$$K(k) := \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}k^{2n} = \frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{1.5}$$

where $0 < k < 1$. The number k is called the modulus of K, and $k' := \sqrt{1-k^2}$ is called the complementary modulus.
Let K, K', L and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l and l', respectively.
Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L}, \tag{1.6}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is induced by (1.6). Following Ramanujan, set $\alpha = k^2$ and $\beta = l^2$. Then we say β is of degree n over α. The multiplier m is defined by

$$m = \frac{K}{L}. \tag{1.7}$$

Let K, K', L_1, L_1', L_2, L_2', L_3 and L_3' denote complete elliptic integrals of the first kind corresponding, in pairs, to the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$, and their complementary moduli, respectively. Let n_1, n_2 and n_3 be positive integers such that $n_3 = n_1 n_2$. Suppose that the equalities

$$n_1\frac{K'}{K} = \frac{L_1'}{L_1}, \quad n_2\frac{K'}{K} = \frac{L_2'}{L_2} \ \text{ and } \ n_3\frac{K'}{K} = \frac{L_3'}{L_3} \tag{1.8}$$

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hold. Then a "mixed" modular equation is a relation between the moduli $\sqrt{\alpha}$, $\sqrt{\beta}$, $\sqrt{\gamma}$ and $\sqrt{\delta}$ that is induced by (1.8). We say that β, γ and δ are of degrees n_1, n_2 and n_3, respectively over α. The multipliers m and m' are associated with α, β and γ, δ.

2 Preliminary results

In this section, we list some relevant identities which are useful in establishing our main results.

Lemma 2.1. [4, Ch. 17, Entry 12 (i) and (iii), p. 124] For $0 < x < 1$, let

$$f(e^{-y}) = \sqrt{z}2^{-1/6}\{x(1-x)e^{y}\}^{1/24}, \tag{2.1}$$

$$f(-e^{-2y}) = \sqrt{z}2^{-1/3}\{x(1-x)e^{y}\}^{1/12}, \tag{2.2}$$

where $z := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$ and $y := \pi \dfrac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}$.

Lemma 2.2. [4, Ch. 16, Entry 24 (ii) and (iv), p. 39] We have

$$f^3(-q) = \varphi^2(-q)\psi(q), \tag{2.3}$$

$$f^3(-q^2) = \varphi(-q)\psi^2(q). \tag{2.4}$$

Lemma 2.3. [4, Ch. 20, Entry 3 (x) and (xi), p. 352] Let β be of degree nine over α and m be the multiplier relating α and β, then

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} = \sqrt{m}, \tag{2.5}$$

$$\left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} = \frac{3}{\sqrt{m}}. \tag{2.6}$$

Lemma 2.4. [5, Ch. 25, Entry 56, p. 210]

$$\text{If} \quad P = \frac{f(-q)}{q^{1/3}f(-q^9)} \quad \text{and} \quad Q = \frac{f(-q^2)}{q^{2/3}f(-q^{18})}, \quad \text{then}$$

$$P^3 + Q^3 = P^2Q^2 + 3PQ. \tag{2.7}$$

Lemma 2.5. [3] If $P = \dfrac{f(-q)}{q^{1/3}f(-q^9)}$ and $Q = \dfrac{f(-q^3)}{qf(-q^{27})}$, then

$$\begin{aligned}(PQ)^3 + \left(\frac{9}{PQ}\right)^3 + 27\left[\left(\frac{P}{Q}\right)^3 + \left(\frac{Q}{P}\right)^3\right] + 243\left(\frac{1}{P^3} + \frac{1}{Q^3}\right) \\ + 9\left(P^3 + Q^3\right) + 81 = \left(\frac{Q}{P}\right)^6.\end{aligned} \tag{2.8}$$

Lemma 2.6. [2] If $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ and $Q = \dfrac{\varphi(q)}{\varphi(q^9)}$, then

$$Q + PQ = 3 + P. \tag{2.9}$$

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Lemma 2.7. [11] If $U := \dfrac{\varphi(-q)}{\varphi(-q^9)}$ and $V := \dfrac{\varphi(-q^2)}{\varphi(-q^{18})}$, then

$$\frac{U}{V} + \frac{V}{U} + 2 = V + \frac{3}{V}. \tag{2.10}$$

3 Modular equations

In this section, we establish some new modular equations of degree 9.

Theorem 3.1. If $U := \dfrac{\varphi(-q)}{\varphi(-q^9)}$ and $V := \dfrac{\varphi(-q^4)}{\varphi(-q^{36})}$, then

$$\begin{aligned} &4\left(U + \frac{2}{U}\right)\left(V + \frac{2}{V}\right) + 3\left(V^2 + \frac{9}{V^2}\right) + \frac{U^2}{V^2} + \frac{V^2}{U^2} + \frac{20}{UV} + 24 \\ &= \left(U + \frac{3}{U}\right)\left[6 + \left(V^2 + \frac{9}{V^2}\right)\right] + 12\left(V + \frac{3}{V}\right). \end{aligned} \tag{3.1}$$

Proof. Replacing q by q^2 in the equation (2.7), we deduce that

$$Y^3 + X^3 = X^2Y^2 + 3XY, \tag{3.2}$$

where

$$X := \frac{f(-q^2)}{q^{2/3} f(-q^{18})} \quad \text{and} \quad Y := \frac{f(-q^4)}{q^{4/3} f(-q^{36})}.$$

Equation (3.2) can be rewritten as

$$a^2 + 3a - W = 0, \tag{3.3}$$

where $a := XY$ and $W := Y^3 + X^3$.
Solving the equation (3.3) for a and then cubing both sides, we deduce that

$$27X^3Y^3 - X^9 + 6X^6Y^3 + 6X^3Y^6 - Y^9 + X^6Y^6 = 0. \tag{3.4}$$

Using the equations (2.3), (2.4) and (2.9) in the equation (3.4), we find that

$$\begin{aligned} &\left(U^2 - 2U^2V + U^2V^2 - 6U + 9 + 4UV - 6V - 2UV^2 + V^2\right)\left(-U^2 + U^2V\right. \\ &\left.+3U - 4UV + 3V - V^2 + UV^2\right)\left(U^4 - U^3V^4 + 4V^3U^3 - 6V^2U^3 + 8VU^3\right. \\ &-9U^3 + 3U^2V^4 - 12U^2V^3 + 24U^2V^2 - 36U^2V + 27U^2 - 3UV^4 + 8UV^3 \\ &\left.-18UV^2 + 36UV - 27U + V^4\right) = 0 \end{aligned} \tag{3.5}$$

By examining the behavior of the above factors of the equation as $q \to 0$, we can find a neighborhood about the origin, where the last factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem the last factor vanishes identically. This completes the proof. Q.E.D.

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Theorem 3.2. If $U := \dfrac{\varphi(-q)}{\varphi(-q^9)}$ and $V := \dfrac{\varphi(-q^6)}{\varphi(-q^{54})}$, then

$$\begin{aligned}
&18\left(2U^2+\frac{13}{U^2}\right)-18\left(4U+\frac{16}{U}\right)-24V^2\left(27U-\frac{1}{U}\right)-\frac{27}{V^3}\left(2U^2+\frac{9}{U^2}\right)\\
&-\left(V^3+\frac{243}{V^3}\right)+\left(4V^2+\frac{243}{V^2}\right)+\left(V+\frac{15}{V}\right)\left[15U-6U^2+U^3\right]+\frac{108U^2}{V^2}\\
&+\frac{108}{U}\left(V+\frac{5}{V}\right)-\frac{9}{U^2}\left(11V+\frac{45}{V}\right)-3\left(17V+\frac{135}{V}\right)+\frac{6}{U^2}\left(4V^2+\frac{81}{V^2}\right)\\
&+\frac{3}{U}\left(V^3+\frac{108}{V^3}\right)-3U^3\left(2+\frac{6}{V^2}-\frac{3}{V^3}\right)-V^3\left(\frac{3}{U^2}+\frac{1}{U^3}\right)+186\\
&+135U\left(\frac{1}{V^3}-\frac{2}{V^2}\right)=0.
\end{aligned} \tag{3.6}$$

Proof. The proof of the equation (3.6) is similar to the proof of the equation (3.1); except that in place of the equation (2.7), we use the equation (2.8). Q.E.D.

4 Mixed modular equations

In this section, we establish several new mixed modular equations akin to those recorded by Ramanujan in his notebooks. Throughout this section, we set

$$A := \frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})}, \quad B_n := \frac{f(-q^n)f(-q^{2n})}{q^n f(-q^{9n})f(-q^{18n})}$$
$$\text{and} \quad C_n := \frac{q^{n/3}f(-q^n)f(-q^{18n})}{f(-q^{2n})f(-q^{9n})}.$$

Theorem 4.1. If $U \neq 1$ and $V \neq 1$, then

$$\frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})} = \frac{U(3-U)}{(1-U)}, \tag{4.1}$$
$$\frac{f(-q)f(-q^2)}{qf(-q^9)f(-q^{18})} = \frac{V(3-V)}{(1-V)}. \tag{4.2}$$

Theorem 4.2. If $V \neq 3$, then

$$\frac{qf^3(-q)f^3(-q^{18})}{f^3(-q^2)f^3(-q^9)} = \frac{U(1-U)}{(3-U)}, \tag{4.3}$$
$$\frac{qf^3(-q)f^3(-q^{18})}{f^3(-q^2)f^3(-q^9)} = \frac{(1-V)}{V(3-V)}, \tag{4.4}$$

where $U := \dfrac{\varphi(-q)}{\varphi(-q^9)}$ and $V := \dfrac{\psi(q)}{q\psi(q^9)}$.

Proof of (4.1) *and* (4.2). Using the equations (2.5) and (2.6), we find that

$$\sqrt{m}\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/8}+1=\frac{3}{\sqrt{m}}+\left\{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right\}^{1/8}. \tag{4.5}$$

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Employing the equations (2.1) and (2.2) in the equation (4.5) and then changing q to $-q$, we arrive at the equation (4.1). By using the equation (2.9) in the equation (4.1), we arrive at the equation (4.2). Q.E.D.

Proofs of (4.3) *and* (4.4). The proofs of the equations (4.3) and (4.4) are similar to the proofs of the equations (4.1) and (4.2), respectively. Hence, we omit the details. Q.E.D.

Theorem 4.3. If $P := AB_2$ and $Q := \dfrac{A}{B_2}$, then

$$Q^2 + \frac{1}{Q^2} = Q + \frac{1}{Q} + \left(\sqrt{P} + \frac{9}{\sqrt{P}}\right)\left(\sqrt{Q} + \frac{1}{\sqrt{Q}}\right) + 6. \tag{4.6}$$

Proof. Using equations (2.3) and (2.9) in the equation (2.7), we deduce that

$$\frac{\varphi^2(-q)}{\varphi^2(-q^9)}\left(\frac{\dfrac{\varphi(-q)}{\varphi(-q^9)} - 3}{\dfrac{\varphi(-q)}{\varphi(-q^9)} - 1}\right) + \frac{\varphi^2(-q^2)}{\varphi^2(-q^{18})}\left(\frac{\dfrac{\varphi(-q^2)}{\varphi(-q^{18})} - 3}{\dfrac{\varphi(-q^2)}{\varphi(-q^{18})} - 1}\right) = A^2 + 3A. \tag{4.7}$$

Using the equation (4.1) in the equation (4.7), we obtain

$$\begin{aligned} &4A^2 - 12A - 24B_2 - 12B_2^2 - 8B_2vu + 8B_2vA + 4B_2^2vA - 4Au + 8B_2v \\ &- 12B_2^2A + 12B_2^2u - 36B_2A + 24B_2u + 12vA + 4B_2^2v + 4B_2A^2 - 4B_2^3A \\ &- 4vA^2 - 4B_2^3 - 4AuB_2 + 4Auv - 4B_2^2vu + 4B_2^3u = 0, \end{aligned} \tag{4.8}$$

where $u := \pm\sqrt{A^2 + 2A + 9}$ and $v := \pm\sqrt{B_2^2 + 2B_2 + 9}$.
Collecting the terms containing u on one side of the equation (4.8) and then squaring both sides, we deduce that

$$\begin{aligned} &vA^2 - 5A^2 + B_2vA^2 - B_2^2A^2 - 2B_2A^2 - B_2^3vA - 3B_2vA + 3B_2^3A \\ &- 2B_2^2vA + B_2^4A + 9B_2^2A + 7B_2A + 36B_2 - 12B_2v - 12B_2^2v + 6B_2^4 \\ &- 5B_2^3v + 21B_2^3 + 40B_2^2 - B_2^4v + B_2^5 = 0. \end{aligned} \tag{4.9}$$

Eliminating v from the equation (4.9) and then setting then $P := AB_2$ and $Q := \dfrac{A}{B_2}$, we arrive at the equation (4.6). Q.E.D.

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Theorem 4.4. If $P := AB_4$ and $Q := \dfrac{A}{B_4}$, then

$$\begin{aligned}
&\left(\sqrt{P}+\frac{3^2}{\sqrt{P}}\right)\left[-170\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)-84\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)-13\left(\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right)\right]\\
&+\left(P+\frac{3^4}{P}\right)\left[-31-4\left(Q^2+\frac{1}{Q^2}\right)-34\left(Q+\frac{1}{Q}\right)\right]+\left(\sqrt{P^3}+\frac{3^6}{\sqrt{P^3}}\right)\\
&\times\left[-6\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)-\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)\right]-\left(P^2+\frac{3^8}{P^2}\right)-365\left(Q+\frac{1}{Q}\right)\\
&-176\left(Q^2+\frac{1}{Q^2}\right)-14\left(Q^3+\frac{1}{Q^3}\right)+\left(Q^4+\frac{1}{Q^4}\right)=782.
\end{aligned}\tag{4.10}$$

Proof. Using the equation (4.1) in the equation (3.1), we deduce that

$$\begin{aligned}
&2592-1440A-2304B_4-864v+864u-736uB_4v-96A^2uB_4^2v-32A^2uB_4^3v\\
&+96A^3B_4^2v+32A^3B_4^3v+480A^2B_4v+288A^2B_4^2v+96A^2B_4^3v+224A^3B_4v\\
&+512A^2uB_4-32A^2uv+448A^2uB_4^2+128A^2uB_4^3+32A^2uB_4^4-64AuB_4^3v\\
&+1888AB_4v+992AB_4^2v+288AB_4^3v+640AuB_4+128Auv+64A^4+608A^2\\
&+704AuB_4^2+256AuB_4^3+64AuB_4^4-416uB_4^2v-96uB_4^3v-192AuB_4^2v\\
&-576Au+224A^2u-2560B_4^2+864B_4v+800B_4^2v+224B_4^3v-224A^2uB_4v\\
&-512A^3B_4-448A^3B_4^2-128A^3B_4^3-32A^3B_4^4-1152A^2B_4+288Av-64A^3u\\
&-96A^2v-1152A^2B_4^2-384A^2B_4^3-96A^2B_4^4-4992AB_4-256AuB_4v\\
&-4032AB_4^2-1280AB_4^3-288AB_4^4+2304uB_4-288uv+1536uB_4^2+32A^3v\\
&+512uB_4^3+96uB_4^4-160A^3-1024B_4^3-224B_4^4=0,
\end{aligned}\tag{4.11}$$

where $u := \pm\sqrt{A^2+2A+9}$ and $v := \pm\sqrt{B_4^2+2B_4+9}$.
Collecting the terms containing u on one side of the equation (4.11) and then squaring both sides, we deduce that

$$\begin{aligned}
&-216AB_4v-68A^2B_4^3v-114A^2B_4^2v-81A^2B_4v-11A^3B_4^2v-28A^2B_4^4v\\
&-288AB_4^2v-181AB_4^3v-A^2B_4^6v-A^3B_4^4v-2AB_4^6v+1458B_4-15AB_4^5v\\
&+2430B_4^2+1998B_4^3-A^4+1034B_4^4+362B_4^5+86B_4^6+13B_4^7+B_4^8-6A^2B_4^5v\\
&-486B_4v-756B_4^2v-558B_4^3v+34A^3B_4+33A^3B_4^2+19A^3B_4^3-68AB_4^4v\\
&+5A^3B_4^4+261A^2B_4+351A^2B_4^2+266A^2B_4^3+116A^2B_4^4+2AB_4^7-10A^3B_4v\\
&+648AB_4+936AB_4^2+673AB_4^3+301AB_4^4-248B_4^4v+38A^2B_4^5-4A^3B_4^3v\\
&+91AB_4^5-A^4B_4+A^4v-70B_4^5v+7B_4^6A^2+17B_4^6A-12B_4^6v\\
&-B_4^7v+A^3B_4^5+A^2B_4^7=0.
\end{aligned}\tag{4.12}$$

Eliminating v from the equation (4.12) and then setting $P := AB_4$ and $Q := \dfrac{A}{B_4}$, we arrive at the equation (4.10). Q.E.D.

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Theorem 4.5. If $P := AB_6$ and $Q := \dfrac{A}{B_6}$, then

$$\begin{aligned}
&\left(\sqrt{P}+\frac{3^2}{\sqrt{P}}\right)\left[162\left(846\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)+243\left(2610\sqrt{Q^3}-\frac{178}{\sqrt{Q^3}}\right)\right.\\
&+243\left(1215\sqrt{Q^5}-\frac{65}{\sqrt{Q^5}}\right)+3\left(116640\sqrt{Q^7}-\frac{763}{\sqrt{Q^7}}\right)+8748\sqrt{Q^{11}}\\
&\left.+6\left(14580\sqrt{Q^9}-\frac{17}{\sqrt{Q^9}}\right)\right]+\left(P+\frac{3^4}{P}\right)\left[23328+81\left(1554Q-\frac{137}{Q}\right)\right.\\
&\left.+27\left(6786Q^2-\frac{286}{Q^2}\right)+729\left(190Q^3-\frac{2}{Q^3}\right)+46656Q^4-\frac{61}{Q^4}+7290Q^5\right]\\
&+\left(\sqrt{P^3}+\frac{3^6}{\sqrt{P^3}}\right)\left[243\left(60\sqrt{Q}-\frac{4}{\sqrt{Q}}\right)+9\left(4164\sqrt{Q^3}-\frac{299}{\sqrt{Q^3}}\right)\right.\\
&\left.+648\left(60\sqrt{Q^5}-\frac{1}{\sqrt{Q^5}}\right)+27\left(684\sqrt{Q^7}-\frac{1}{\sqrt{Q^7}}\right)+3888\sqrt{Q^9}\right]\\
&+\left(P^2+\frac{3^8}{P^2}\right)\left[729+27\left(176Q-\frac{36}{Q}\right)+27\left(300Q^2-\frac{8}{Q^2}\right)+1539Q^4\right.\\
&\left.+6\left(864Q^3-\frac{1}{Q^3}\right)\right]+\left(\sqrt{P^5}+\frac{3^{10}}{\sqrt{P^5}}\right)\left[\left(38\sqrt{Q}+\frac{13}{\sqrt{Q}}\right)+532\sqrt{Q^7}\right.\\
&\left.+\left(270\sqrt{Q^3}+\frac{54}{\sqrt{Q^3}}\right)+\left(1080\sqrt{Q^5}+\frac{27}{\sqrt{Q^5}}\right)\right]+9\left(186057Q^3-\frac{2518}{Q^3}\right)\\
&+\left(P^3+\frac{3^{12}}{P^3}\right)\left[-13+\left(90Q-\frac{9}{Q}\right)+144Q^2\right]+4374\left(351Q^2-\frac{22}{Q^2}\right)\\
&+\left(\sqrt{P^7}+\frac{3^{14}}{\sqrt{P^7}}\right)\left[\frac{1}{\sqrt{Q}}+12\sqrt{Q^3}+12\sqrt{Q^5}\right]+\left(P^4+\frac{3^{16}}{P^4}\right)(Q^2)\\
&+972\left(1539Q-\frac{143}{Q}\right)+81\left(8100Q^4-\frac{277}{Q^4}\right)+9\left(11664Q^5-\frac{7}{Q^5}\right)\\
&+\left(6561Q^6+\frac{1}{Q^6}\right)+373491=0.
\end{aligned} \tag{4.13}$$

Proof. The proof of the equation (4.13) is similar to the proof of the equation (4.10). So we omit the details. Q.E.D.

Theorem 4.6. If $P := C_1C_2$ and $Q := \dfrac{C_1}{C_2}$, then

$$\left(\sqrt{P}+\frac{1}{\sqrt{P}}\right)\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)=\left(P+\frac{1}{P}\right). \tag{4.14}$$

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Proof. Using the equation (4.3) in the equation (2.10), we deduce that

$$\begin{aligned}&3u-3v-2C_1^3u-C_2^3v+6C_1^3C_2^3-C_1^3v+uC_2^6-6uC_2^3+uv-uC_2^3v\\&+C_1^3C_2^3v+2C_1^6-1+C_2^6-C_1^3C_2^6-13C_1^3-2C_2^3=0,\end{aligned} \tag{4.15}$$

where $u := \pm\sqrt{C_1^2-10C_1+1}$ and $v := \pm\sqrt{C_2^2-10C_2+1}$.
Collecting the terms containing u on one side of the equation (4.15) and then squaring both sides, we deduce that

$$\begin{aligned}&2C_2^3v-33C_1^3C_2^3+9C_1^3v+40C_1^3C_2^6+2C_1^6-7C_2^6+9C_1^3-2C_2^3-12C_1^3C_2^3v\\&+C_1^6C_2^3v-C_2^9C_1^3v+8C_2^6C_1^3v-C_2^6v-13C_2^9C_1^3-C_2^6C_1^6+C_2^12C_1^3+C_2^9\\&-2vC_1^6+7C_1^6C_2^3=0.\end{aligned} \tag{4.16}$$

Collecting the terms containing v on one side of the equation (4.16) and then squaring both sides, we arrive at

$$A(q)B(q)=0, \tag{4.17}$$

where $A(q)=C_2C_1^4+C_1^3-C_1C_2-C_1^3C_2^3+C_2^3+C_2^4C_1$ and

$$\begin{aligned}B(q)=&\,C_2^2C_1^8+C_2^4C_1^7-C_2C_1^7+C_2^6C_1^6-2C_1^6C_2^3+2C_2^5C_1^5-2C_2^2C_1^5\\&+C_2^7C_1^4-3C_2^4C_1^4+C_2C_1^4-2C_1^3C_2^6+2C_1^3C_2^3+C_1^6+C_2^8C_1^2\\&-2C_2^5C_1^2+C_2^2C_1^2-C_2^7C_1+C_2^4C_1+C_2^6.\end{aligned} \tag{4.18}$$

Consider the sequence $\{q_n\}=\left\{\dfrac{1}{n+1}\right\}$, $n=1,2,3\ldots$, which has a limit in $|q|<1$. We see $A(q_n)=0 \quad \forall n$, where as $B(q_n)\neq 0 \quad \forall n$. Then by zeros of analytic functions, $A\equiv 0$ on $|q|<1$ [8]. By setting $P:=C_1C_2$ and $Q:=\dfrac{C_1}{C_2}$, we arrive at (4.14). This completes the proof. Q.E.D.

Theorem 4.7. If $P:=C_1C_4$ and $Q:=\dfrac{C_1}{C_4}$, then

$$\begin{aligned}&\left(\sqrt{P^3}+\frac{1}{\sqrt{P^3}}\right)\left[5\left(\sqrt{Q}+\frac{1}{\sqrt{Q}}\right)+4\left(\sqrt{Q^3}+\frac{1}{\sqrt{Q^3}}\right)+\left(\sqrt{Q^5}+\frac{1}{\sqrt{Q^5}}\right)\right]\\&+14\left(Q+\frac{1}{Q}\right)+7\left(Q^2+\frac{1}{Q^2}\right)+\left(Q^3+\frac{1}{Q^3}\right)+12=\left(P^3+\frac{1}{P^3}\right).\end{aligned} \tag{4.19}$$

Proof. The proof of the equation (4.19) is similar to the proof of the equation (4.14); except that in place of the equation (2.10), we use the equation (3.1). Q.E.D.

5 A remarkable product of theta–functions

Mahadeva Naika, Maheshkumar and Sushan Bairy [17], have introduced a new remarkable product of theta–functions as in the equation (1.4). They have also established some new general formulas for the explicit evaluations of $b_{s,t}$.

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Recently, Mahadeva Naika, Chandankumar and Hemanthkumar[13], have established several new modular identities connecting the remarkable product of theta–functions $b_{s,9}$ with $b_{r^2s,9}$ for $r = 3$, 5 and 11 and also established some new values for $b_s, 9$.

In this section, we establish several new modular identities connecting the remarkable product of theta–functions $b_{s,9}$ with $b_{r^2s,9}$ for $r = 2$, 4 and 6.

Lemma 5.1. [17] If s and t are any positive rational, then

$$b_{2s,t}b_{\frac{2}{s},t} = 1. \tag{5.1}$$

Lemma 5.2. [15] We have, $0 < b_{s,t} \leq 1$ for all $s \geq 2$ and t positive integer greater than 1.

Theorem 5.3. If $X := \sqrt{b_{s,9}b_{4s,9}}$ and $Y := \sqrt{\dfrac{b_{4s,9}}{b_{s,9}}}$, then

$$Y^2 + \frac{1}{Y^2} = Y + \frac{1}{Y} + 3\left(\sqrt{X} + \frac{1}{\sqrt{X}}\right)\left(\sqrt{Y} + \frac{1}{\sqrt{Y}}\right) + 6. \tag{5.2}$$

Proof. Using the equation (4.6) along with the equation (1.4) with $t := 9$, we arrive at (5.2). Q.E.D.

Corollary 5.4. We have

$$b_{1,9} = \left(\frac{\sqrt{3}+1}{2} + \frac{3^{1/4}}{\sqrt{2}}\right)^2, \tag{5.3}$$

$$b_{4,9} = \left(\frac{\sqrt{3}+1}{2} - \frac{3^{1/4}}{\sqrt{2}}\right)^2. \tag{5.4}$$

Proofs of (5.3) *and* (5.4). Putting $s = 1$ in the equation (5.2) and using the fact that $b_{1,9}b_{4,9} = 1$, we deduce that

$$\left(h^4 - 2h^3 - 2h + 1\right)\left(h^2 + h + 1\right)^2 = 0, \tag{5.5}$$

where $h := \sqrt{b_{4,9}}$.

We observe that the first factor of the equation (5.5) vanishes and other factors does not vanish for the specific value of $q = e^{-\pi\sqrt{4/9}}$. Hence, we have

$$u^2 - 2u - 2 = 0, \tag{5.6}$$

where $u := \sqrt{b_{4,9}} + \dfrac{1}{\sqrt{b_{4,9}}}$.

On solving the equation (5.6) and $u > 0$ by Lemma (5.2) , we find that

$$\sqrt{b_{4,9}} + \frac{1}{\sqrt{b_{4,9}}} = 1 + \sqrt{3}. \tag{5.7}$$

On solving the equation (5.7), we arrive at (5.3) and (5.4). Q.E.D.

Remark 5.5. Another proof of $b_{1,9}$ and $b_{4,9}$ can be found in [13].

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Theorem 5.6. If $X := \sqrt{b_{s,9}b_{16s,9}}$ and $Y := \sqrt{\dfrac{b_{16s,9}}{b_{s,9}}}$, then

$$\begin{aligned}&3\left(\sqrt{X}+\frac{1}{\sqrt{X}}\right)\left[-170\left(\sqrt{Y}+\frac{1}{\sqrt{Y}}\right)-84\left(\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right)-13\left(\sqrt{Y^5}+\frac{1}{\sqrt{Y^5}}\right)\right]\\&+9\left(X+\frac{1}{X}\right)\left[-31-4\left(Y^2+\frac{1}{Y^2}\right)-34\left(Y+\frac{1}{Y}\right)\right]+27\left(\sqrt{X^3}+\frac{1}{\sqrt{X^3}}\right)\\&\times\left[-6\left(\sqrt{Y}+\frac{1}{\sqrt{Y}}\right)-\left(\sqrt{Y^3}+\frac{1}{\sqrt{Y^3}}\right)\right]-81\left(X^2+\frac{1}{X^2}\right)-365\left(Y+\frac{1}{Y}\right)\\&-176\left(Y^2+\frac{1}{Y^2}\right)-14\left(Y^3+\frac{1}{Y^3}\right)+\left(Y^4+\frac{1}{Y^4}\right)=782.\end{aligned} \tag{5.8}$$

Proof. Using the equation (4.10) along with the equation (1.4) with $t := 9$, we arrive at (5.8). Q.E.D.

Corollary 5.7. We have

$$b_{8,9}=\frac{(\sqrt{3}-\sqrt{2})^2(\sqrt{3}-1)^2}{2}, \tag{5.9}$$

$$b_{1/2,9}=\frac{(\sqrt{3}+\sqrt{2})^2(\sqrt{3}+1)^2}{2}. \tag{5.10}$$

Proofs of (5.9) *and* (5.10). Putting $s = 1/2$ in the equation (5.8) and using the fact that $b_{1/2,9}b_{8,9} = 1$, we deduce

$$\left(h^4-4h^3-12h^2-4h+1\right)\left(h^2+h+1\right)^2\left(h^4+h^3+3h^2+h+1\right)^2=0, \tag{5.11}$$

where $h := \sqrt{b_{8,9}}$.
We observe that the first factor of the equation (5.11) vanishes and other factors does not vanish for the specific value of $q = e^{-\pi\sqrt{8/9}}$. Hence, we have

$$u^2-4u-14=0, \tag{5.12}$$

where $u := \sqrt{b_{8,9}} + \dfrac{1}{\sqrt{b_{8,9}}}$.
On solving the equation (5.12) and $u > 0$, we find that

$$\sqrt{b_{8,9}}+\frac{1}{\sqrt{b_{8,9}}}=2+3\sqrt{2}. \tag{5.13}$$

On solving the equation (5.13), we arrive at (5.9) and (5.10). Q.E.D.

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Theorem 5.8. If $X := \sqrt{b_{s,9}b_{36s,9}}$ and $Y := \sqrt{\dfrac{b_{36s,9}}{b_{s,9}}}$, then

$$\begin{aligned}
&3\left(\sqrt{X}+\frac{1}{\sqrt{X}}\right)\Bigg[162\left(846\sqrt{Y}+\frac{1}{\sqrt{Y}}\right)+243\left(2610\sqrt{Y^3}-\frac{178}{\sqrt{Y^3}}\right)\\
&+243\left(1215\sqrt{Y^5}-\frac{65}{\sqrt{Y^5}}\right)+3\left(116640\sqrt{Y^7}-\frac{763}{\sqrt{Y^7}}\right)+8748\sqrt{Y^{11}}\\
&+6\left(14580\sqrt{Y^9}-\frac{17}{\sqrt{Y^9}}\right)\Bigg]+9\left(X+\frac{1}{X}\right)\Bigg[23328+81\left(1554Y-\frac{137}{Y}\right)\\
&+27\left(6786Y^2-\frac{286}{Y^2}\right)+729\left(190Y^3-\frac{2}{Y^3}\right)+\left(46656Y^4-\frac{61}{Y^4}\right)+7290Y^5\Bigg]\\
&+27\left(\sqrt{X^3}+\frac{1}{\sqrt{X^3}}\right)\Bigg[243\left(60\sqrt{Y}-\frac{4}{\sqrt{Y}}\right)+9\left(4164\sqrt{Y^3}-\frac{299}{\sqrt{Y^3}}\right)\\
&+648\left(60\sqrt{Y^5}-\frac{1}{\sqrt{Y^5}}\right)+27\left(684\sqrt{Y^7}-\frac{1}{\sqrt{Y^7}}\right)+3888\sqrt{Y^9}\Bigg]\\
&+81\left(X^2+\frac{1}{X^2}\right)\Bigg[729+27\left(176Y-\frac{36}{Y}\right)+27\left(300Y^2-\frac{8}{Y^2}\right)+1539Y^4\\
&+6\left(864Y^3-\frac{1}{Y^3}\right)\Bigg]+3^5\left(\sqrt{X^5}+\frac{1}{\sqrt{X^5}}\right)\Bigg[\left(38\sqrt{Y}+\frac{13}{\sqrt{Y}}\right)+532\sqrt{Y^7}\\
&+\left(270\sqrt{Y^3}+\frac{54}{\sqrt{Y^3}}\right)+\left(1080\sqrt{Y^5}+\frac{27}{\sqrt{Y^5}}\right)\Bigg]+9\left(186057Y^3-\frac{2518}{Y^3}\right)\\
&+3^6\left(X^3+\frac{1}{X^3}\right)\left[-13+\left(90Y-\frac{9}{Y}\right)+144Y^2\right]+4374\left(351Y^2-\frac{22}{Y^2}\right)\\
&+3^7\left(\sqrt{X^7}+\frac{1}{\sqrt{X^7}}\right)\left[\frac{1}{\sqrt{Y}}+12\sqrt{Y^3}+12\sqrt{Y^5}\right]+3^8\left(X^4+\frac{1}{X^4}\right)(Y^2)\\
&+972\left(1539Y-\frac{143}{Y}\right)+81\left(8100Y^4-\frac{277}{Y^4}\right)+9\left(11664Y^5-\frac{7}{Y^5}\right)\\
&+\left(6561Y^6+\frac{1}{Y^6}\right)+373491=0.
\end{aligned} \tag{5.14}$$

Proof. Using the equation (4.13) along with the equation (1.4) with $t := 9$, we arrive at (5.14). Q.E.D.

Corollary 5.9. We have

$$b_{12,9} = \frac{\left((5-3\sqrt{3})\,x^2+x-1\right)^2}{9}, \tag{5.15}$$

$$b_{1/3,9} = \left(x^2+x\sqrt{3}+2+\sqrt{3}\right)^2, \tag{5.16}$$

where $x := (5+3\sqrt{3})^{1/3}$.

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Proofs of (5.15) *and* (5.16). Setting $s = 1/3$ in the equation (5.14) and using the fact that $b_{1/3,9}b_{12,9} = 1$, we deduce that

$$\begin{aligned}&\left(9h^6+18h^5+27h^4+6h^3-3h^2-12h+1\right)\left(3h^3+3h^2+3h+1\right)^2\\&\left(9h^6+18h^5+27h^4+21h^3+12h^2+3h+1\right)^2=0,\end{aligned}\tag{5.17}$$

where $h := \sqrt{b_{12,9}}$.
We observe that the first factor of the equation (5.17) vanishes and other factors does not vanish for the specific value of $q = e^{-\pi\sqrt{12/9}}$. Hence, we have

$$\left(9h^6+18h^5+27h^4+6h^3-3h^2-12h+1\right)=0.\tag{5.18}$$

On solving the equation (5.18) by using Maple and $0 < h < 1$, we arrive at (5.15) and (5.16). Q.E.D.

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